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SLOPE INEQUALITIES FOR FIBRED SURFACES VIA GIT

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Abstract

In this paper we present a generalisation of a theorem due to Cornalba and Harris, which is an application of Geometric Invariant Theory to the study of invariants of fibrations. In particular, our generalisation makes it possible to treat the problem of bounding the invariants of general fibred surfaces. As a first application, we give a new proof of the slope inequality and of a bound for the invariants associated to double cover fibrations.

Introduction

In [23] and [9], Xiao and Cornalba-Harris developed two methods that can be applied to the problem of bounding the invariants of fibred varieties. Given a complex variety X fibred over a curve, the starting point of both methods is a line bundle L on X. However, while Xiao's method uses techniques of vector bundle stability, the one of Cornalba-Harris exploits Geometric Invariant Theory (GIT). In the same papers, the three authors treat the case of surfaces fibred over a curve, proving a fundamental inequality on the invariants: the so-called slope inequality (cf. Section 2). However, Cornalba and Harris prove the inequality only for semistable fibrations (i.e. fibred surfaces such that any fibre is a semistable curve in the sense of Deligne and Mumford). In fact, their method applies only to semistable non-hyperelliptic fibrations, and the semistable hyperelliptic case is obtained by an ad hoc argument.

A result of Tan [21, 22] made apparent that the general case of the slope inequality cannot be reduced to the semistable case (see Remark 2.2).

The starting point of this work is the question whether or not it is possible to treat the non-semistable and the hyperelliptic cases using the ideas of Cornalba-Harris. The answer is affirmative; while the extension to non-semistable fibrations is straightforward, in order to treat the hyperelliptic case it is necessary to modify substantially the method. This led us to develop a generalisation of the method which is, in our opinion, interesting on its own.

Until now, the method of Xiao have been almost the only one used to find lower bounds on the slope of fibred surfaces (although the very nice argument introduced by

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Moriwaki in [16] should also be taken into account, as remarked also in [2]). It has been further developed and applied by Ohno, Konno, Barja, Zucconi, and others.

Thanks to the generalisation presented here, the Cornalba-Harris method qualifies as a valid alternative. In this paper, besides proving the slope inequality, we can show in a new and direct way that the fibred surfaces realising the equality are all hyperelliptic. Furthermore, we prove an inequality holding between the invariants of double cover fibrations.

In [4] both the generalised Cornalba-Harris method and the one of Xiao have been applied, obtaining a new bound on the slope of non-Albanese fibrations. Besides, the first method seems to have promising applications to the case of fibrations of higher dimensional varieties [5], where the one of Xiao tends to be technically hard.

The importance of these kind of results is double. On the one hand, they are fundamental tools in the study of the geography of complex surfaces (for example, Pardini's recent proof of the Severi inequality in [18], makes an essential use of the slope inequality). On the other hand, the bounds on the slope have an application to the positivity of divisors on the moduli space of stable curves of genus g (for instance, in [12], the slope inequality is a key ingredient for attaching a conjecture on the nef cone).

We now give a brief account of the method of Cornalba-Harris and of its generalisation. The idea of the method is the following. Let $f\colon X \to T$ be a flat proper morphism of complex varieties with a line bundle L on X whose restrictions to the general fibres of f give embeddings in projective spaces. Suppose that the Hilbert points of these embeddings are semistable in the sense of GIT. Then the key-point of the method is to translate the semistability assumption into the existence of a line bundle on the base T, together with a non-vanishing section of it. This produces in particular an element in the effective cone of the base T. When T is a curve, the consequence is a non-trivial inequality holding between the degrees of rational classes of divisors on it.

The main point of the generalisation is to drop the assumption that the line bundle L gives an *embedding* on the general fibres, and to consider *arbitrary rational maps*. In order to do this, we need to introduce a suitable generalisation of Hilbert (semi)stability for a variety with a map in a projective space (Definitions 1.1 and 1.3). This generalisation sounds unexpected because, as GIT is mainly used to construct moduli spaces, GIT stability is usually defined for line bundles whose associated morphisms encode all the information about the variety, as in the case of the classical Hilbert points. We prove that, assuming this generalised semistability, the argument of Cornalba-Harris, with some modifications, can be pushed through, and still gives as a consequence an effective divisor on the base T (Theorem 1.5).

The paper is organised as follows. In the first section we prove the main theorem (Theorem 1.5); under some more restrictive assumptions, we can derive explicit inequalities on the rational classes of divisors on the base (Corollary 1.6). In Section 2, we give the proof of the slope inequality, and of the fact that the fibrations with minimal slope are all hyperelliptic (Proposition 2.4). We treat in Section 3 the case of surfaces having an involution on a genus γ fibration (double cover fibrations, see Definition 3.1), proving an inequality on the invariants.

1. The Cornalba-Harris method generalised

Let G be a reductive complex algebraic group and V a finite dimensional representation of G. An element $v \in V$ is said to be GIT *semistable* if the closure of its orbit does not contain 0, and GIT *stable* if its stabiliser is finite and its orbit closed. Recall that a necessary and sufficient condition for the semistability of $v \in V$ is the existence of a G-invariant non-constant homogeneous polynomial $f \in \mathrm{Sym}(V^\vee)$ such that $f(v) \neq 0$.

Let X be a variety (an integral separated scheme of finite type over $\mathbb{C}$), with a linear system $V \subseteq H^0(X, L)$, for some line bundle L on X. Fix $h \geq 1$ and call G_h the image of the natural homomorphism

$$\mathrm{Sym}^h V \xrightarrow{\varphi_h} H^0(X, L^h). \tag{1.1}$$

Set $N_h = \dim G_h$ and take exterior powers

$$\bigwedge^{N_h} \mathrm{Sym}^h V \xrightarrow{\bigwedge^{N_h} \varphi_h} \bigwedge^{N_h} G_h = \det G_h.$$

If we identify $\det G_h$ with $\mathbb{C}$, the homomorphism $\bigwedge^{N_h} \varphi_h$ can be seen as a liner functional on $\bigwedge^{N_h} \mathrm{Sym}^h V$. Changing the isomorphism, it gets multiplied by a non-zero element of $\mathbb{C}$. Hence, we can see $\bigwedge^{N_h} \varphi_h$ as a well-defined element of $\mathbb{P}\big(\bigwedge^{N_h} \mathrm{Sym}^h V^\vee\big)$.

DEFINITION 1.1. With the above notations, we call $\bigwedge^{N_h} \varphi_h \in \mathbb{P}\big(\bigwedge^{N_h} \mathrm{Sym}^h V^\vee\big)$, the generalised h-th Hilbert point associated to the couple (X, V).

If V induces an embedding, then for $h \gg 0$ the homomorphism φ_h is surjective and it is the classical h-th Hilbert point associated to ψ.

Let $\dim V = s + 1$ and consider the standard representation $SL(s+1, \mathbb{C}) \to SL(V)$; we get an induced natural action of $SL(s+1, \mathbb{C})$ on $\mathbb{P}\big(\bigwedge^{N} \mathrm{Sym}^h V^\vee\big)$, and we can introduce the associated notion of GIT (semi)stability: we say that the h-th generalised Hilbert point of the couple (X, V) is *semistable* (*resp. stable*) *if it is GIT semistable* (*resp. stable*) *with respect to the natural* $SL(s+1, \mathbb{C})$*-action.*

REMARK 1.2. Let (X, V) be as above. Consider the factorization of the induced map through the image,

$$X \dashrightarrow \overline{X} \overset{j}{\hookrightarrow} \mathbb{P}^s.$$

Set $\overline{L} = j^*(\mathcal{O}_{\mathbb{P}^s}(1))$ and let $\overline{V} \subseteq H^0(\overline{X}, \overline{L})$ be the linear systems associated to j. The homomorphism (1.1) factors as follows:

$$\mathrm{Sym}^h V \cong \mathrm{Sym}^h \overline{V} \xrightarrow{\overline{\varphi}_h} H^0(\overline{X}, \overline{L}^h) \hookrightarrow H^0(X, L^h),$$

where the homomorphism $\overline{\varphi}_h$ is the h-th Hilbert point of the embedding j; notice that, by Serre's vanishing theorem, this homomorphism is onto (and, in particular, $G_h = H^0(X, \overline{L}^h)$) for large enough h. The generalised h-th Hilbert point of (X, V) is therefore naturally identified with the h-th Hilbert point of $(\overline{X}, \overline{V})$, and the generalised Hilbert stability of (X, V) coincides with the classical Hilbert stability of the embedding j.

DEFINITION 1.3. We say that (X, V) is generalised Hilbert stable (resp. semistable) if its generalised h-th Hilbert point is stable (resp. semistable) for infinitely many integers $h > 0$.

In the case of embeddings in projective space, this notion coincides with the classical Hilbert stability introduced in [17].

1.1. The theorem. We will use the following well known fact about vector bundles and representations.

REMARK 1.4. Let T be a projective variety. Consider a vector bundle E of rank r on T and a complex holomorphic representation

$$GL(r, \mathbb{C}) \xrightarrow{\rho} GL(V).$$

Composing the transition functions of E with ρ, we can construct a new vector bundle, which we call E_ρ. Hence, if $\{g_{\alpha,\beta}\}$ is a system of transition functions for E with respect to an open cover $\{\mathcal{U}_{\alpha,\beta}\}$ of T, then a system of transition functions for E_ρ with respect to the same cover is $\{\rho(g_{\alpha,\beta})\}$. Clearly E_ρ has typical fibre V.

For instance, if we consider as ρ the representation corresponding to symmetric, tensor and exterior power, the vector bundle E_ρ becomes respectively $\mathrm{Sym}^n E$, $\bigotimes^n E$ and $\bigwedge^n E$.

We are now ready to state the theorem. Notation: given a sheaf $\mathcal{F}$ over a variety T, we call $\mathcal{F} \otimes \boldsymbol{k}(t)$ the fibre of $\mathcal{F}$ over the point $t \in T$.

Theorem 1.5. *Let $f : X \to T$ be a flat morphism from a variety X to a variety T. Let t be a general point of T, X_t the fibre of f at t. Let L be a line bundle on X and $\mathcal{F}$ a locally free subsheaf of f_*L of rank r. Suppose that for some integer $h > 0$ the h-th generalised Hilbert point associated to the linear system $\mathcal{F} \otimes \boldsymbol{k}(t) \subseteq$*

$H^0(X_t, L_{|X_t})$ is semistable. Let $\mathcal{G}_h \subseteq f_*L^h$ be a locally free subsheaf that contains the image of the morphism

$$\operatorname{Sym}^h \mathcal{F} \to f_*L^h,$$

and coincides with it at t. Set $N_h = \operatorname{rank} \mathcal{G}_h$. Let $\mathcal{L}_h$ be the line bundle

$$\mathcal{L}_h = \det(\mathcal{G}_h)^r \otimes (\det \mathcal{F})^{-hN_h}.$$

Then there is a positive integer m, depending only on h, $\operatorname{rank}\mathcal{F}$ and N_h, such that $(\mathcal{L}_h)^m$ is effective.

Proof. In what follows, t is a general point of T. Set $F := \mathcal{F} \otimes \boldsymbol{k}(t)$, $G_h := \mathcal{G}_h \otimes \boldsymbol{k}(t)$. Consider the morphism $\operatorname{Sym}^h \mathcal{F} \xrightarrow{\gamma_h} \mathcal{G}_h$. Its fibre at t, $\bar{\gamma}_h \colon \operatorname{Sym}^h F \to G_h$, is surjective by assumption. Its maximal exterior power is the generalised Hilbert point associated to (X_t, F). Therefore there exists by assumption a homogeneous $SL(F)$-invariant polynomial (of degree, say, d) $P \in \operatorname{Sym}^d\left(\bigwedge^{N_h} \operatorname{Sym}^h F\right)$ such that

$$\operatorname{Sym}^d \bigwedge^{N_h} \bar{\gamma}_h(P) \neq 0 \quad \text{in} \quad (\det G_h)^d. \tag{1.2}$$

We may assume (simply taking a power of P if necessary) that the degree of P is mr, where m is an integer depending only on h, r and N_h. Fixing an isomorphism $F \cong \mathbb{C}^r$, P corresponds to an element

$$\tilde{P} \in \operatorname{Sym}^{mr} \bigwedge^{N_h} \operatorname{Sym}^n \mathbb{C}^r.$$

If we change the isomorphism, as P is invariant by the action of $SL(F)$, we obtain $\tilde{P}$ multiplied by a non-zero element of $\mathbb{C}$. Hence, the line W generated by $\tilde{P}$ in $\operatorname{Sym}^{mr}\left(\bigwedge^{N_h} \operatorname{Sym}^h \mathbb{C}^r\right)$, is well defined and invariant under the action of $GL(r, \mathbb{C})$. This produces naturally a line bundle on T with an injective morphism into $(\det \mathcal{G}_h)^{mr}$, as we verify at once, using the language of representations.

Let ρ be the N_h-th exterior power of the h-th symmetric power of the standard representation,

$$\rho \colon GL(r, \mathbb{C}) \to GL\left(\bigwedge^{N_h} \operatorname{Sym}^h \mathbb{C}^r\right).$$

Using the notations of Remark 1.4, the vector bundle $\mathcal{F}_\rho$ is $\bigwedge^{N_h} \operatorname{Sym}^h \mathcal{F}$. Let

$$\sigma \colon GL(r, \mathbb{C}) \to GL(W)$$

be the representation obtained by restriction from $\mathrm{Sym}^{mr}\rho$. Thus, there is an inclusion of vector bundles $\mathcal{F}_\sigma \hookrightarrow \mathrm{Sym}^{mr}\mathcal{F}_\rho$. Composing this inclusion with $\mathrm{Sym}^{mr}\bigwedge^{N_h}\gamma_h$, we obtain a homomorphism $\mathcal{F}_\sigma \to (\det\mathcal{G}_h)^{mr}$, whose fibre at t is the following composition

$$W \hookrightarrow \mathrm{Sym}^{mr}\bigwedge^{N}\mathrm{Sym}^n(F) \to (\det G_h)^{mr},$$

which is a non-zero homomorphism by construction because of property (1.2) (it is, roughly speaking, the evaluation of γ_h on P). It remains to understand explicitly $\mathcal{F}_\sigma$. Given an element $M \in GL(r,\mathbb{C})$, if we write $M = (\det M)^{1/r}U$, where $U \in SL(r,\mathbb{C})$, the action of M on P is the following:

$$\sigma(M)P = \mathrm{Sym}^{mr}\rho((\det M)^{1/r}U)P = \det\mu(M)^{hNm}\,\mathrm{Sym}^{mr}\rho(U)P = \det\mu(M)^{hNm}P.$$

It follows that in our case $\mathcal{F}_\sigma$ is the line bundle $(\det\mathcal{F})^{hNm}$, and the proof is concluded. $\square$

In all the applications of the Cornalba-Harris method that have been made so far, including ours, the condition of stability is satisfied not for a *fixed* h, but for h *large enough*: more precisely Hilbert stability is satisfied (see Definition 1.3).

Moreover, it is often the case that the choice of $\mathcal{F} \subseteq f_*L$ and of $\mathcal{G}_h$ is such that the first rational Chern class $c_1(\mathcal{L}_h) \in A^1(T)_{\mathbb{Q}}$ is a polynomial in h of the form

$$c_1(\mathcal{L}_h) = \alpha_d h^d + \cdots + \alpha_1 h + \alpha_0, \quad \alpha_i \in A^1(T)_{\mathbb{Q}}. \tag{1.3}$$

Theorem 1.5 assures that for infinitely many positive integers h there exists an integer m such that the line bundle $\mathcal{L}_h^m$ is effective, hence the class $c_1(\mathcal{L}_h) \in A^1(T)_{\mathbb{Q}}$ is effective. In this situation, we can therefore conclude that the leading coefficient α_d is the limit in $A^1(T)_{\mathbb{Q}}$ of effective divisors.[1]

We can make explicit computations and simplifications under additional assumptions (this corollary should be compared to the original Theorem (1.1) of [9]).

Corollary 1.6. *With the notations of* Theorem 1.5, *suppose that $\mathcal{F}$ induces a Hilbert semistable map on the general fibres. Suppose moreover that*
(1) *f is proper, T is irreducible of dimension k and X is of pure dimension $k+d$;*
(2) *for $t \in T$ general, the fibre $\mathcal{F} \otimes \boldsymbol{k}(t)$ induces an embedding of X_t;*
(3) *the higher direct images $R^i f_* L^h$ vanish for $i > 0$, $h \gg 0$ (this happens for instance if the fibre of $\mathcal{F}$ induces an ample linear system on* any *fibre of f).*

[1] Note that, although we are speaking of limits, we don't need to pass to real coefficients, because in fact both α_d and the members of the succession converging to it given by (1.3) belong to $A^1(T)_{\mathbb{Q}}$.

Then, the class

$$\mathcal{E}(L,\mathcal{F}) := r f_*(c_1(L)^{d+1}\cap[X]) - (d+1)c_1(\mathcal{F})\cap f_*(c_1(L)^d\cap[X])$$

is contained in the closure of the effective cone of $A_{k-1}(T)_{\mathbb{Q}}$.

Proof. By the second assumption, for general t, the homomorphism

$$\mathrm{Sym}^h\mathcal{F}\otimes\boldsymbol{k}(t)\to H^0(X_t, L^h_{|X_t})$$

is surjective for large enough h. Hence, we can choose $\mathcal{G}_h = f_*L^h$ in Theorem 1.5. Therefore,

$$c_1(\mathcal{L}_h) = r c_1(f_*L^h) - h \operatorname{rank} f_*L^h c_1(\mathcal{F}).$$

The first assumption enables us to use the Riemann-Roch theorem for singular varieties ([11], Corollary 18.3.1) and obtain the formula

$$\mathrm{ch}(f_! L^h\cap \mathrm{td}(\mathcal{O}_T)) = f_*(\mathrm{ch}(L^h)\cap\mathrm{td}(\mathcal{O}_X)). \tag{1.4}$$

Recalling that, for any variety Y, $\mathrm{td}(\mathcal{O}_Y) = [Y] +$terms of dimension $< \dim Y$, and using standard intersection-theoretical computations, we obtain that

$$\begin{aligned} c_1(\mathcal{L}_h)\cap[T] = {} & \frac{h^{d+1}}{(d+1)!}\mathcal{E}(L,\mathcal{F}) \\ & + \sum_{i=1}^{d}(-1)^{i+1}(r c_1(R^i f_*L^h)\cap[T] - h\operatorname{rank}(R^i f_*L^h)c_1(\mathcal{F})\cap[T]) \\ & + O(h^d). \end{aligned} \tag{1.5}$$

Hence, equation (1.5), together with the remarks made above, implies the statement. □

2. Bounds on the slope of fibred surfaces

A *fibred surface*, is the datum of a surjective morphism f with connected fibres from a smooth projective surface X to a smooth complete curve B. Throughout this section, we shall use the term "fibration" as a synonym of fibred surface. The genus g of the general fibre is called genus of the fibration. We call a fibration *relatively minimal* if the fibres contain no -1-curves. A fibration is said to be *semistable* if all the fibres are semistable curves in the sense of Deligne and Mumford (i.e. if it is relatively minimal with nodal fibres). From any fibred surface $f\colon X\to B$, by contracting all the -1-curves in the fibres, we obtain an induced fibration on B, called the relatively minimal model of f, which is unique if $g\geq 1$. We say that a fibration is *locally trivial* if it is a holomorphic fibre bundle.

As usual, the *relative canonical sheaf* of a fibred surface $f : X \to B$ is the line bundle $\omega_f = \omega_X \otimes (f^*\omega_B)^{-1}$; and let K_f denote any associated divisor. From now on we will consider *relatively minimal fibrations of genus* $g \geq 2$. Two basic invariants for such a fibration are the following.

$$K_f^2 = K_X^2 - 8(g-1)(g(B)-1);$$
$$\chi_f = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_B)\chi(\mathcal{O}_F) = \chi(\mathcal{O}_X) - (g-1)(g(B)-1).$$

Using Riemann-Roch and Leray's spectral sequence, one sees that $\chi_f = \deg f_*\omega_f$. It is well known that both these invariants are non-negative. Moreover, $\chi_f = 0$ if and only if f is locally trivial. Assuming that the fibration is not locally trivial, we can consider the ratio

$$s(f) := \frac{K_f^2}{\chi_f},$$

which is called the *slope*. Of course $s(f) \geq 0$; but a bigger bound holds, given by the following result, which we call *slope inequality*:

Theorem 2.1 (Xiao, Cornalba-Harris). *Let $f : X \to B$ be a relatively minimal fibration of genus $g \geq 2$.*

$$gK_f^2 \geq 4(g-1)\chi_f. \tag{2.1}$$

This inequality is sharp, and it is possible to classify the fibrations reaching it, which are in particular all hyperelliptic (Proposition 2.4).

REMARK 2.2. As is well-known, the process of *semistable reduction* associates to any fibred surface a semistable one, by means of a ramified base change. One might hope that, using semistable reduction, it could be possible to reduce the proof of the slope inequality for any fibration to the semistable case. However, Tan has shown (cf. Theorem A and Theorem B of [21, 22]) that the behaviour of the slope under base change *cannot be controlled when the base change ramifies over fibres which are not D-M semistable*, which is precisely what happens in the semistable reduction process. In particular, the inequalities that can be shown to hold for semistable fibrations, do not necessarily extend to arbitrary fibrations.

The form of Theorem 1.5 we shall use in the applications to surfaces is the following.

Corollary 2.3. *Let $f\colon X \to B$ be a fibred surface. Let L be a line bundle on X and $\mathcal{F}$ a coherent*[2] *subsheaf of f_*L of rank r such that for general $b \in B$ the linear system*

$$\mathcal{F} \otimes \boldsymbol{k}(b) \subseteq H^0(X_b, L_{|X_b})$$

induces a Hilbert semistable map. Let $\mathcal{G}_h$ be a coherent subsheaf of f_*L^h that contains the image of the morphism $\mathrm{Sym}^h\mathcal{F} \to f_*L^h$, and coincides with it at general b. If $N = \mathrm{rank}\,\mathcal{G}_h$ is of the form $Ah + O(1)$ and $\deg \mathcal{G}_h$ of the form $Bh^2 + O(h)$, the following inequality holds:

$$rB - A \deg \mathcal{F} \geq 0. \tag{2.2}$$

Proof. Straightforward from Theorem 1.5 and the observations made after it. □

We now come to the proof of the slope inequality.

Proof of Theorem 2.1. We want to apply Corollary 2.3 with $L = \omega_f$ and $\mathcal{F} = f_*L$. Let X_b be a general fibre.

Observe that the higher direct image $R^1 f_*\omega_f^h$ vanishes for large enough h, as can be seen for instance using the relative version of Kawamata-Viehweg vanishing theorem (cf. [13], Theorem 1.2.3). We split the proof in two steps:

(1) *Suppose f is non-hyperelliptic*. The condition of Corollary 2.3 is satisfied, because the canonical embedding of a smooth non-hyperelliptic curve is Hilbert stable, as shown in [17]: indeed (using Mumford's notations), it is linearly stable, and hence Chow stable, which in turns implies the generalised Hilbert stability; see also [1] or [20] for a direct proof. We can compute the terms in inequality (2.2) as follows

$$\begin{aligned}
&\mathrm{rank}\ f_*\omega_f = h^0(X_b, \omega_{f\,|X_b}) = g;\\
&\mathrm{rank}\,\mathcal{G}_h = h^0(X_b, \omega_{f\,|X_b}^{h}) = (2h-1)(g-1);\\
&\deg \mathcal{G}_h = \frac{(hK_f \cdot (h-1)K_f)}{2} + \deg\ f_*\omega_f = h^2\frac{K_f^2}{2} + O(h).
\end{aligned}$$

Hence, inequality (2.2) becomes exactly the slope inequality.

(2) *Suppose f is hyperelliptic*. A general hyperelliptic fibred surface is not always a double cover of a fibration of genus 0. Anyway we show below that for our purposes it can be treated as if it were. We make use of a standard argument (cf. for instance [2]) which can be applied to any fibred surface with an involution that restrict to an involution on the general fibres.

First observe that the hyperelliptic involution on the general fibres extends to a global involution ι on X (see for instance [19]). If ι has no isolated fixed points then

[2] As the base B is a smooth curve, any coherent subsheaf of a locally free sheaf is locally free.

$X/\langle\iota\rangle$ is a smooth genus 0 fibred surface over B and the quotient map is a double cover whose ramification divisor is the fixed locus of ι. Otherwise, we blow up the isolated fixed points and obtain a smooth surface $\tilde{X}$ birational to X whose induced involution $\tilde{\iota}$ has no isolated fixed points. Call Y the quotient of $\tilde{X}$ by $\tilde{\iota}$. The surface Y has a natural genus 0 fibration α over B, but is not necessarily relatively minimal. We have the following diagram:

$$\begin{tikzcd} \tilde{X} \arrow[d, "\eta"] \arrow[dr, "\pi"] & \\ X \arrow[r, dashed] \arrow[d, "f"] & Y \arrow[dl, "\alpha"] \\ B & \end{tikzcd} \tag{2.3}$$

Let $R \subset Y$ be the branch divisor of π. By the theory of cyclic coverings (cf. [7] I.17), we can find a line bundle $\mathcal{L}$ on Y such that $\mathcal{L}^2 = \mathcal{O}_Y(R)$. Set $\tilde{f} = f \circ \eta$. Recall that $\omega_{\tilde{f}} = \eta^*\omega_f \otimes \mathcal{O}_{\tilde{X}}(E)$, where E is the union of the exceptional -1-curves. Let ϵ be the number of connected components of E. Consider the exact sequence

$$0 \to \eta^*\omega_f^h \to \omega_{\tilde{f}}^h \to \mathcal{O}_{hE}(hE) \to 0$$

and the long exact sequence induced by the pushforward by $\tilde{f}$:

$$\begin{aligned} 0 \to\ & f_*\omega_f^h \to \tilde{f}_*\omega_{\tilde{f}}^h \to \tilde{f}_*\mathcal{O}_{hE}(hE) \\ & \to R^1 f_*\omega_f^h \to R^1 \tilde{f}_*\omega_{\tilde{f}}^h \to R^1 \tilde{f}_*\mathcal{O}_{hE}(hE) \to 0. \end{aligned}$$

Observe that $\deg \tilde{f}_*\mathcal{O}_{hE}(hE) = h^0(\mathcal{O}_{hE}(hE)) = 0$, and that

$$\deg R^1 \tilde{f}_*\mathcal{O}_{hE}(hE) = h^1(\mathcal{O}_{hE}(hE)) = \epsilon\frac{h^2-h}{2},$$

by the Riemann-Roch Theorem for embedded curves. Therefore $\tilde{f}_*\omega_{\tilde{f}}^h = f_*\omega_f^h$ for any h, and

$$\deg R^1 \tilde{f}_*\omega_{\tilde{f}}^h = \deg R^1 f_*\omega_f^h + \epsilon\frac{h^2-h}{2} = \epsilon\frac{h^2-h}{2}.$$

Recall that in our situation $\omega_{\tilde{f}} = \pi^*(\omega_\alpha \otimes \mathcal{L})$ and $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_Y \oplus \mathcal{L}^{-1}$. Therefore we have the following decomposition of $\tilde{f}_*\omega_{\tilde{f}}$

$$\tilde{f}_*\omega_{\tilde{f}} = \alpha_*\pi_*\omega_{\tilde{f}} = \alpha_*(\pi_*\pi^*(\omega_\alpha\otimes\mathcal{L})) = \alpha_*((\omega_\alpha\otimes\mathcal{L})\otimes\pi_*\mathcal{O}_Y) = \alpha_*(\omega_\alpha\otimes\mathcal{L})\oplus\alpha_*\omega_\alpha. \tag{2.4}$$

Hence, $f_*\omega_f = \alpha_*(\omega_\alpha \otimes \mathcal{L})$, being α a genus 0 fibration.

The canonical line bundle $\omega_{X_b} = \omega_{f\,|X_b}$ induces a morphism to $\mathbb{P}^{g-1}$ that factors through a double cover of $\mathbb{P}^1$ ramified at the Weierstrass points of X_b composed with the Veronese embedding of degree $g-1$. The morphism $\mathrm{Sym}^h f_*\omega_f \to f_*\omega_f^h$ has fibre on b

$$\mathrm{Sym}^h H^0(X_b,\omega_{X_b}) = \mathrm{Sym}^h H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(g-1)) \twoheadrightarrow H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(h(g-1))) \subset H^0(X_b,\omega_{X_b}^h).$$

Observe that the fibre $\alpha_*(\omega_\alpha\otimes\mathcal{L})^h\otimes \boldsymbol{k}(b)$ is $H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(h(g-1)))$; we hence choose $\alpha_*(\omega_\alpha\otimes\mathcal{L})^h$ as the sheaf $\mathcal{G}_h$ in Corollary 2.3. The semistability assumption is satisfied, because the Veronese embedding $\mathbb{P}^1\hookrightarrow\mathbb{P}^{g-1}$ has semistable Hilbert point, as shown for instance in [14], Corollary 5.3. For large enough h, by the Riemann-Roch theorem

$$\deg\mathcal{G}_h = h^2\frac{(K_\alpha+L)^2}{2} + \deg R^1\alpha_*(\omega_\alpha\otimes\mathcal{L})^h + O(h),$$
$$\operatorname{rank}\mathcal{G}_h = h^0(Y_b,\omega_{Y_b}^h(hL)) = h(g-1)+1.$$

We now estimate the degree of $R^1\alpha_*(\omega_\alpha\otimes\mathcal{L})^h$ for $h\gg 0$. Observe that $R^1\tilde{f}_*\omega_{\tilde{f}}^h$ is torsion and splits into the direct sum

$$R^1\tilde{f}_*\omega_{\tilde{f}}^h = R^1\alpha_*(\omega_\alpha\otimes\mathcal{L})^h \oplus R^1\alpha_*(\omega_\alpha^h\otimes\mathcal{L}^{h-1}).$$

Now, observe that for large enough h

$$\deg R^1\alpha_*(\omega_\alpha\otimes\mathcal{L})^h = \deg R^1\alpha_*(\omega_\alpha^h\otimes\mathcal{L}^{h-1}) + O(h),$$

hence

$$\deg R^1\alpha_*(\omega_\alpha\otimes\mathcal{L})^h = \frac{1}{2}\deg R^1\tilde{f}_*\omega_{\tilde{f}}^h + O(h) = \epsilon\frac{h^2}{4} + O(h),$$

and inequality (2.2) becomes

$$\frac{g}{2}\left((K_\alpha+L)^2+\frac{\epsilon}{2}\right) - (g-1)\deg\alpha_*(\omega_\alpha\otimes\mathcal{L}) \geq 0.$$

As π is a finite morphism of degree 2, and η is a sequence of ϵ blow ups,

$$K_f^2 - \epsilon = K_{\tilde{f}}^2 = (\pi^*(K_\alpha+L))^2 = 2(K_\alpha+L)^2.$$

Remembering that $\deg\alpha_*(\omega_\alpha\otimes\mathcal{L}) = \deg\tilde{f}_*\omega_{\tilde{f}} = \deg f_*\omega_f = \chi_f$, we obtain the slope inequality. □

The following proposition has been proved by Konno in [15], some years later the proof of the slope inequality, as a by-product of other inequalities. Using the approach of Cornalba-Harris, it is a natural consequence of the construction. This proposition is a generalisation of the first part of Theorem (4.12) of [9].

Proposition 2.4. *Let $f : X \to B$ be a relatively minimal non-locally trivial fibred surface of genus ≥ 2 satisfying equality in* Theorem 2.1. *Then f is hyperelliptic.*

Proof. Suppose by contradiction that it holds $gK_f^2 = 4(g-1)\chi_f$ and that f is non-hyperelliptic. Going back to the proof of Theorem 1.5, we see that in order to have equality, the degree of the degree 2 coefficient in the polynomial $c_1(\mathcal{L}_h)$ must be 0. As $c_1(\mathcal{L}_h)$ is effective, the linear coefficient has to be of non-negative degree. Computing this class we get

$$0 \leq -\frac{g}{2}K_f^2 + (g-1)\chi_f = -(g-1)\chi_f,$$

which is strictly negative for $g \geq 2$ and f non-locally trivial. Hence, we get the desired contradiction. □

The hyperelliptic fibrations that reach the bound can be classified, and turn out to have restrictions on the type of singularities of the special fibres (see [9], Theorem (4.12) for the semistable case, and [2] Section 2.2 for the general one).

3. Bounds for double cover fibrations

Arguing in a very similar way to what we did for hyperelliptic fibrations, we can prove a bound for the invariants of a more general class of fibred surfaces, double cover fibrations:

DEFINITION 3.1. A *double cover fibration of type* (g, γ) is the data of a genus g fibred surface $f : X \to B$ together with a global involution on X that restricts, on the general fibre, to an involution with genus γ quotient.

In particular, the double cover fibrations of type $(g, 0)$ are exactly the hyperelliptic ones. The slope of double cover fibrations has been studied in [6], and recently in [10]. We refer to these two articles for a detailed discussion of the situation. In [10] the sharp bound

$$s(f) \geq 4\frac{g-1}{g-\gamma} \tag{3.1}$$

is proved, under the assumption $g \geq 4\gamma + 1$. For $g < 4\gamma$ the bound is *false* in general. Proposition 3.2 below implies that the bound holds in general for a particular class of double cover fibrations. A similar inequality can be found applying Xiao's method ([3], Proposition 4.10).

Let $f : X \to B$ be a double cover fibration of type (g, γ) with $\gamma \geq 1$. With the same construction made for the hyperelliptic case, we can associate to it a genus γ fibration $\alpha : Y \to B$, not necessarily relatively minimal, obtaining a diagram of the form (2.3). Let us use the same notations of the hyperelliptic case.

Proposition 3.2. *Let* $f: X \to B$ *be a double cover fibration of type* (g, γ) *with* $\gamma \geq 1$ *and* $g \geq 2\gamma + 1$. *Let* $\alpha: Y \to B$ *be the associated fibration of genus* γ *described above.*

Then the following inequality holds:

$$K_f^2 \geq 4\frac{g-1}{g-\gamma}(\chi_f - \chi_\alpha). \tag{3.2}$$

In particular, any double cover fibration with $g \geq 2\gamma + 1$ *and associated genus* γ *fibration isotrivial satisfies the bound* (3.1).

Proof. Arguing as in the hyperelliptic case (same notations), we obtain the decomposition

$$f_*\omega_f = \alpha_*(\omega_\alpha \otimes \mathcal{L}) \oplus \alpha_*\omega_\alpha,$$

which on a general fibre X_b amounts to

$$H^0(X_b, \omega_{X_b}) = H^0(Y_b, \omega_{Y_b}(L)) \oplus H^0(Y_b, \omega_{Y_b}),$$

where L is the restriction of $\mathcal{L}$ to Y_b. By Hurwitz' formula $\deg L = g - 2\gamma + 1$. We want to apply Corollary 2.3 of Theorem 1.5 to the rank $g-\gamma$ subsheaf $\mathcal{F} := \alpha_*(\omega_\alpha \otimes \mathcal{L})$ of $f_*\omega_f$.

We split the proof in two cases.

(1) Suppose that the restriction of $\mathcal{F}$ on a general fibre Y_b does not belong to a g_2^1 on Y_b (this holds in particular if α is non-hyperelliptic or if $g \geq 2\gamma+2$). In this case $\mathcal{F}$ induces on a general fibre X_b a $2:1$ morphism to Y_b followed by the morphism ψ in $\mathbb{P}^{g-\gamma-1}$ induced by the line bundle $\omega_{Y_b}(L)$. We distinguish again two cases. (1.a) ψ is an embedding; in this case it is linearly stable, by [17], Section 2.15, hence, by the same argument made in the non-hyperelliptic case of Theorem 2.1, it is Hilbert stable. We apply Corollary 2.3 taking as $\mathcal{G}_h$ the sheaf $\alpha_*(\omega_\alpha^h \otimes \mathcal{L}^h)$. Now, computing $\deg \mathcal{G}_h$, $\operatorname{rank} \mathcal{G}_h$, and $\deg R^1\alpha_*(\omega_\alpha \otimes \mathcal{L})^h$ for $h \gg 0$, as in the hyperelliptic case of Theorem 2.1, inequality (2.2) becomes

$$\frac{g-\gamma}{2}\left((K_\alpha + L)^2 + \frac{\epsilon}{2}\right) - (g-1)\deg \alpha_*(\omega_\alpha \otimes \mathcal{L}) \geq 0.$$

Remembering that

$$K_f^2 - \epsilon = K_{\tilde{f}}^2 = \pi^*(K_\alpha + L)^2 = 2(K_\alpha + L)^2,$$

and that $\deg \alpha_*(\omega_\alpha \otimes \mathcal{L}) = \deg \tilde{f}_*\omega_{\tilde{f}} - \deg \alpha_*\omega_\alpha = \deg f_*\omega_f - \deg \alpha_*\omega_\alpha = \chi_f - \chi_\alpha$, we obtain the statement. (1.b) ψ fails to be an embedding if and only if $\deg L = 2$. Note that, by assumption, if C is hyperelliptic, $L \notin g_2^1$. In this case ψ is a birational morphism, which is linearly semistable, and hence, by [17] again, its image is Chow semistable. Chow semistability does not imply Hilbert semistability, hence we cannot use the Cornalba-Harris method; however, we can in this case apply a result of Bost ([8], Theorem 3.3) that gives as a consequence exactly the same inequality of Corollary 2.3.

(2) Suppose on the other hand that α is hyperelliptic and that the morphism induced by $\alpha_*\omega_\alpha \otimes \mathcal{L}$ on a general fibre factors through the hyperelliptic involution of Y_b:

$$X_b \xrightarrow{2:1} Y_b \xrightarrow{2:1} \mathbb{P}^1 \overset{v}{\hookrightarrow} \mathbb{P}^{g-\gamma-1},$$

where v is the Veronese embedding. The semistability assumption is satisfied because v is Hilbert semistable (as observed in the hyperelliptic case in Theorem 2.1) with similar computations, we obtain

$$\deg \mathcal{G}_h = h^2 \frac{K_f^2}{8} + O(h), \quad \operatorname{rank} \mathcal{G}_h = \gamma h + O(1),$$

and again inequality (2.2) gives the desired bound. □

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